CHAPTER 3

TRIGONOMETRIC MEASUREMENTS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Measure angles in degrees, radians, and mils.
2. Find angular velocity and the area of a sector using radians.
3. Apply the Pythagorean theorem and properties of similar right triangles to problem solving.
4. Apply trigonometric ratios, functions, and tables to problem solving.

INTRODUCTION

This is the first of several chapters in this course dealing with the subject of trigonometry. Chapters 4, 5, and 6 also deal directly with triangles and trigonometry. Chapter 7 deals with vectors and forces. The study of vectors and forces is so closely related to trigonometry that it is normally included in a trigonometry course.

Mathematics, Volume 1, introduces numerical trigonometry and some applications in problem solving. However, trigonometry is not restricted to solving problems involving triangles; it also forms a foundation for some advanced mathematical concepts and subject areas. Trigonometry is both algebraic and geometric in nature, and in this course both of these qualities will be applied.

MEASURING ANGLES

Mathematics, Volume 1, pointed out that an angle is formed when two straight lines intersect. In this course, an angle is considered to be generated when a line having a set direction is rotated about a point, as depicted in figure 3-1.

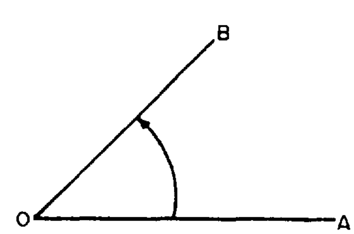

Figure 3-1.—Generation of an angle.

In figure 3-1, line OA is laid out as a reference line having a set direction. One end of the line is used as a pivot point and the line is rotated from its initial position (line OA) to another position (line OB), as in opening a door. As the line turns on its pivot point, it generates the angle AOB.

The following terminology is used in this and subsequent chapters:

1. *Radius vector*—The line that is rotated to generate an angle.
2. *Initial position*—The original position of the radius vector; corresponds to line OA in figure 3-1.
3. *Terminal position*—The final position of the radius vector; corresponds to line OB in figure 3-1.
4. *Positive angle*—The angle generated by rotating the radius vector counterclockwise from the initial position.
5. *Negative angle*—The angle generated by rotating the radius vector clockwise from the initial position.

The convention of identifying angles by use of Greek letters is followed in this text. When only one angle is involved, it will be symbolized by θ (theta). Other Greek letters will be used when more than one angle is involved. The additional symbols used will be ϕ (phi), α (alpha), and β (beta).

DEGREES

The degree system is the most common system of angular measurement. In this system a complete revolution is divided into 360 equal parts called *degrees*; so,

$$1 \text{ revolution} = 360°$$

For accuracy, each degree is divided into 60 minutes; so,

$$1° = 60'$$

Each minute is divided into 60 seconds; so,

$$1' = 60''$$

For convenience in working with angles, the 360° are divided into four equal parts of 90° each, similar to the rectangular coordinate system. The 90° sectors, called *quadrants*, are numbered according to the convention shown in figure 3-2.

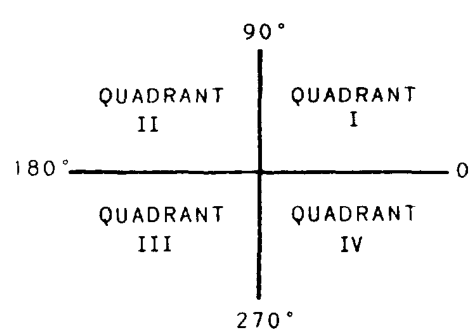

Figure 3-2.—Quadrant positions.

If the angle generated by rotating the radius vector in a positive (counterclockwise) direction is between 0° and 90°, then the angle is in the first quadrant. If the angle is between 90° and 180°, then the angle is in the second quadrant. If the angle is between 180° and 270°, then the angle is in the third quadrant. And if the angle is between 270° and 360°, then the angle is in the fourth quadrant.

If the angle generated by rotating the radius vector in a positive direction is more than 360°, then the quadrant in which the angle lies is found by subtracting from the angle the largest multiple of 360° that the angle contains. The quadrant in which the remainder angle lies is determined as described in the previous paragraph. *The original angle lies in the same quadrant as the remainder angle.*

EXAMPLE: In which quadrant is the angle 130°?

SOLUTION: Since 130° is between 90° and 180°, it is in the second quadrant. (See fig. 3-3, view A).

EXAMPLE: In which quadrant is the angle 850°?

SOLUTION: The largest multiple of 360° contained in 850° is 720°; so, 850° − 720° = 130°. Since 130° is in the second quadrant, then 850° also lies in the second quadrant. This relationship is shown in figure 3-3, view B.

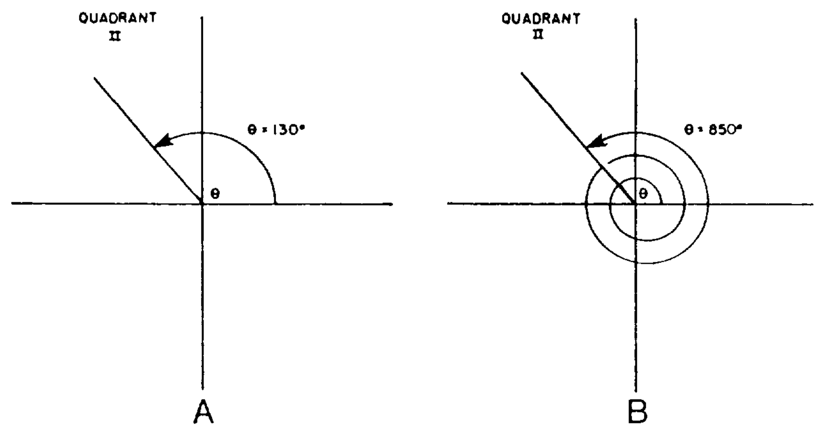

Figure 3-3.—Angle generation.

PRACTICE PROBLEMS:

Determine the quadrant in which each of the following angles lies:

1. 260°

2. 290°

3. 800°

4. 1,930°

ANSWERS:

1. 3rd

2. 4th

3. 1st

4. 2nd

RADIANS

Another even more fundamental method of angular measurement involves the *radian*. It has certain advantages over the degree method. Radian measurement greatly simplifies work with trigonometric functions in calculus. Radian measurement also relates the length of arc generated to the size of an angle.

A *radian* is defined as an angle that, if its vertex is placed at the center of a circle, intercepts an arc equal in length to the radius vector of the circle. Assume that an angle is generated, as shown in figure 3-4, view A. If we impose the condition that the length of the arc, *s*, described by the extremity of the line segment generating the angle, must

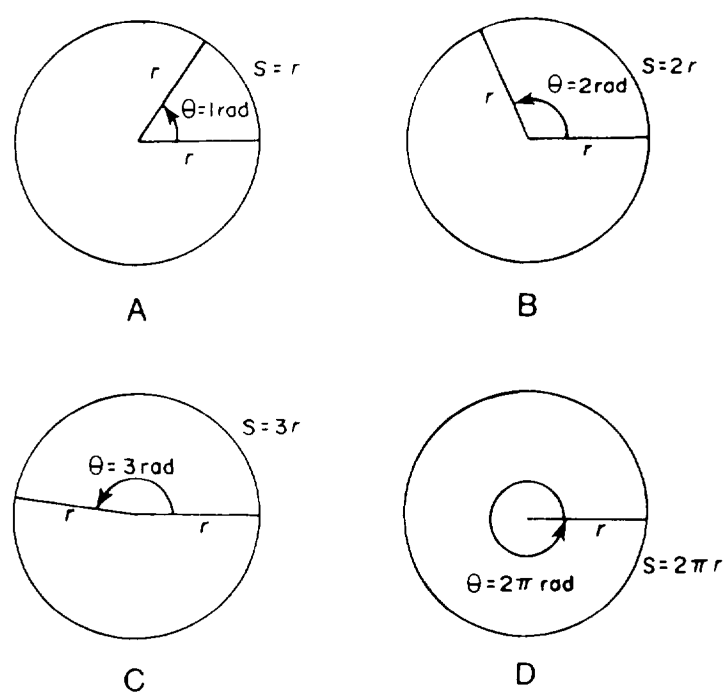

Figure 3-4.—Radian measure.

equal the length of the radius vector, r, then we would describe an angle exactly one radian in size; that is, for 1 radian,

$$s = r$$

In a broader sense, *the radian measure of an angle, θ, is the ratio of the length of the arc, s, it subtends to the length of the radius vector, r, of the circle in which it is the central angle*; that is,

$$\theta = \frac{s}{r}$$

For angle θ, in figure 3-4, view B, which intercepts an arc equal to two times the length of the radius vector, θ equals two radians. For angle θ, in figure 3-4, view C, which intercepts an arc equal to three times the length of the radius vector, θ equal three radians.

EXAMPLE: Find the radian measure of the central angle in a circle with a radius of 10 inches if the angle subtends an arc of 5 inches.

SOLUTION:

$$\theta = \frac{s}{r}$$

$$= \frac{5}{10}$$

$$= 0.5 \text{ radians}$$

Recall from plane geometry that the circumference of a circle is 2π times the radius or

$$C = 2\pi r$$

Hence, the radius vector can be laid off on the circumference 2π times. (See fig. 3-4, view D).

Since the arc length of the circumference is 2π radians and the circumference encompasses one complete revolution of 360°, then

$$2\pi \text{ radians} = 360^\circ$$

One-half of a revolution equals 180° or π radians; so,

$$\pi \text{ radians} = 180^\circ \qquad (3.1)$$

By dividing both sides of equation (3.1) by π, we find that

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$= 57.2958^\circ \text{ (rounded)}$$

$$= 57^\circ\ 17'\ 45''$$

By dividing both sides of equation (3.1) by 180, we find that

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

$$= 0.01745 \text{ radians (rounded)}$$

NOTE: The degree symbol (°) is customarily used to indicate degrees, and a pure number with no symbol attached is used to indicate radians. For example, sin 3 should be understood to represent "sine of 3 radians," whereas the "sine of 3 degrees" would be written sin 3°.

The following list indicates other relationships frequently used in trigonometric problems:

Radians	Degrees
$\pi/6$	30
$\pi/4$	45
$\pi/3$	60
$\pi/2$	90
π	180
$3\pi/2$	270
2π	360

EXAMPLE: Express 160° in radians, using π in the answer.

SOLUTION:

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

$$160^\circ = 160 \times 1^\circ$$

$$= 160 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{8\pi}{9} \text{ radians}$$

EXAMPLE: Express $\pi/20$ in degrees.

SOLUTION:

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$\frac{\pi}{20} \text{ radians} = \frac{\pi}{20} \times 1 \text{ radian}$$

$$= \frac{\pi}{20} \times \frac{180^\circ}{\pi}$$

$$= \frac{180^\circ}{20}$$

$$= 9^\circ$$

Refer to figure 3-5. We can see that if θ represents the number of radians in a central angle, r the length of the radius of the circle, and s the length of the intercepted arc, then the *length of the arc* equals the number of radians multiplied by the length of the radius or

$$s = \theta r$$

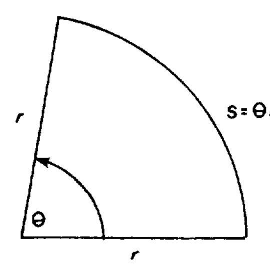

Figure 3-5.—Length of arc.

EXAMPLE: In a circle having a radius of 11 inches, an arc subtends a central angle of 3 radians. Find the length of the arc in inches.

SOLUTION:

$$s = \theta r$$

$$= 3 \cdot 11$$

$$= 33 \text{ inches}$$

PRACTICE PROBLEMS:

1. Find the number of radians in the central angle subtended by an arc 18 inches long in a circle whose radius is 8 inches.

Express the following angles in radians, using π in the answer:

2. $420°$

3. $135°$

Express the following angles in degrees:

4. 20π

5. $5\pi/6$

6. In a circle whose radius, r, is 4 inches, find in inches the length of arc, s, whose central angle is 1 1/4 radians.

ANSWER:

1. 9/4 radians

2. $7\pi/3$

3. $3\pi/4$

4. $3{,}600°$

5. $150°$

6. 5 inches

Because of the relationship of the radian to arc length, the radian has some special applications in measurements of angular velocity and area of a sector.

Angular Velocity

Another type of problem that radian measurement simplifies is that which relates the rotating motion of the wheels of a vehicle to its forward motion. Here we will not be dealing with angles alone but also with *angular velocity*. Let's analyze this type of motion.

Consider the circle at the left in figure 3-6 to indicate the original position of a wheel. As the wheel turns, it rolls so that the center moves along the line CC', where C' is the center of the wheel at its final position. The contact point at the bottom of the wheel moves an equal distance PP'; but as the wheel turns through angle θ, arc s is made to coincide with line PP'; so,

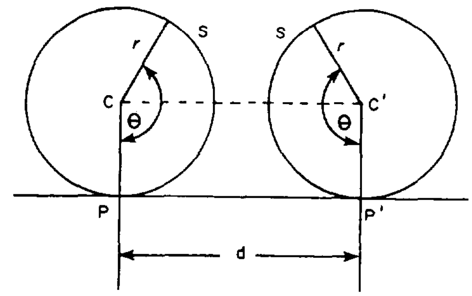

Figure 3-6.—Angular rotation.

$$s = PP' = d$$

or the length of arc is equal to the forward distance, d, the wheel travels. But since

$$s = r\theta$$

then the forward distance that the wheel travels is

$$d = r\theta$$

Dividing both sides of the previous equation by t gives

$$\frac{d}{t} = \frac{r\theta}{t}$$

When a vehicle moves with a constant velocity, v, in time, t, the distance, d, the vehicle travels is expressed by the formula

$$d = vt$$

Solving this formula for v, we have

$$v = \frac{d}{t}$$

The fraction d/t expresses the *linear velocity* of the vehicle, and θ/t is the *angular velocity*. If we let ω (Greek letter omega) stand for the angular velocity, then the equation

$$\frac{d}{t} = \frac{r\theta}{t}$$

becomes

$$v = r\omega$$

where ω is measured in radians per unit time.

EXAMPLE: A car wheel is rotating at 1,050 revolutions per minute (rpm). Find

1. the angular velocity in radians per second.
2. the linear velocity in meters per second on the tire tread, 25 centimeters from the center.

SOLUTION:

1. To find the angular velocity, we need to convert rev/min to rad/sec. To do this, we will apply unit conversions (multiples of one) as follows:

$$\omega = 1{,}050 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{1}{60} \frac{\text{min}}{\text{sec}}$$

$$= \frac{(1{,}050)(2\pi)}{60} \frac{\text{rad}}{\text{sec}}$$

$$= 35\pi \text{ radians per second}$$

2. We find the linear velocity as follows:

$$v = r\omega$$

$$= 25 \text{ cm} \times 35\pi \frac{\text{rad}}{\text{sec}}$$

$$= 875\pi \frac{\text{cm}}{\text{sec}}$$

NOTE: When no unit of angular measure is indicated, the angle is understood to be expressed in radians.

We now need to convert cm/sec to m/sec. We will again apply a unit conversion:

$$v = 875\pi \frac{\text{cm}}{\text{sec}} \times \frac{1}{100} \frac{\text{m}}{\text{cm}}$$

$$= \frac{875\pi}{100} \frac{\text{m}}{\text{sec}}$$

$$= 8.75\pi \text{ meters per second}$$

EXAMPLE: A car is traveling 40 miles per hour. If the wheel radius is 16 inches, what is the angular velocity of the wheels in

1. radians per minute?
2. revolutions per minute?

SOLUTION:

1. We know that

$$v = r\omega$$

Thus,

$$\omega = \frac{v}{r}$$

$$= \frac{40}{16}\ \frac{\text{mi/hr}}{\text{in}}$$

$$= \frac{5}{2}\ \frac{\text{mi}}{\text{hr} \times \text{in}} \times \frac{5{,}280}{1}\ \frac{\text{ft}}{\text{mi}} \times \frac{12}{1}\ \frac{\text{in}}{\text{ft}} \times \frac{1}{60}\ \frac{\text{hr}}{\text{min}}$$

$$= \frac{(5)(5{,}280)(12)}{(2)(60)}\ \frac{\text{rad}}{\text{min}}$$

$$= 2{,}640 \text{ radians per minute}$$

2. Since 2π radians $= 360°$ and $360° = 1$ revolution, then

$$\omega = 2{,}640\ \frac{\text{rad}}{\text{min}} \times \frac{1}{2\pi}\ \frac{\text{rev}}{\text{rad}}$$

$$= \frac{2{,}640}{2\pi}\ \frac{\text{rev}}{\text{min}}$$

$$= 420.2 \text{ revolutions per minute}$$

EXAMPLE: Determine the distance a truck will travel in 1 minute if the wheels are 3 feet in diameter and are turning at the rate of 5 revolutions per second. HINT: Diameter $= 2 \times$ radius

SOLUTION:

$$v = r\omega$$

$$\frac{d}{t} = r\omega$$

$$d = rt\omega$$

$$= \frac{3}{2} \text{ ft} \times 1 \text{ min} \times \left(5\ \frac{\text{rev}}{\text{sec}} \times \frac{2\pi}{1}\ \frac{\text{rad}}{\text{rev}}\right)$$

$$= \frac{3}{2} \text{ ft} \times 1 \text{ min} \times 10\pi\ \frac{\text{rad}}{\text{sec}} \times \frac{60}{1}\ \frac{\text{sec}}{\text{min}}$$

$$= \frac{(3)(10\pi)(60)}{2} \text{ ft}$$

$$= 2{,}827.43 \text{ feet}$$

Area of a Sector

From plane geometry we find that the area of the sector of a circle is proportional to the angle enclosed in the sector.

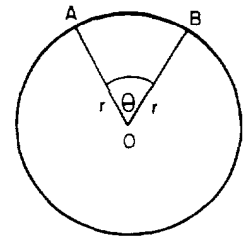

Figure 3-7.—Sector of a circle.

Consider sector AOB of the circle shown in figure 3-7. If θ is increased to 2π radians (360°), it encompasses the entire circle; so the area of the circle is proportional to 2π radians. Hence,

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

But the area of a circle can be found by the formula

$$A = \pi r^2$$

By substitution, we find

$$\text{area of sector} = \frac{\theta}{2\pi}(\pi r^2)$$

$$= \frac{\theta r^2}{2}$$

Therefore, the *area of a sector of a circle* can be found by the formula

$$A = \frac{1}{2} r^2 \theta$$

where θ is expressed in radians.

EXAMPLE: Find the area of a sector of a circle with a radius of 6 inches having a central angle of 60°.

SOLUTION:

$$A = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2}(6 \text{ in})^2\left(60^\circ \times \frac{\pi}{180^\circ}\right)$$

$$= \frac{36\pi}{(2)(3)} \text{ in}^2$$

$$= 6\pi \text{ square inches}$$

The area of a sector of a circle can also be found if the radius and arc length are known. Since

$$s = r\theta$$

then

$$A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}r(r\theta)$$

$$= \frac{1}{2}rs$$

EXAMPLE: What is the diameter of a circle if a sector of the circle has an arc length of 9 inches and an area of 18 square inches?

SOLUTION:

If

$$A = \frac{1}{2}rs$$

then

$$r = \frac{2A}{s}$$

$$= \frac{2(18 \text{ in}^2)}{9 \text{ in}}$$

$$= 4 \text{ in}$$

But

$$d = 2r$$

Therefore,

$$d = 2(4 \text{ in})$$

$$= 8 \text{ inches}$$

PRACTICE PROBLEMS:

1. A car travels 4,500 feet in 1 minute. The diameter of the wheels is 36 inches. What is the angular velocity of the wheels in radians per minute?

2. How far in feet does a car travel in 1 minute if the radius of the wheels is 18 inches and the angular velocity of the wheels is 1,000 radians per minute?

3. Find the area of a sector of a circle whose central angle is $\pi/3$ and whose diameter is 24 inches. Leave the answer in terms of π.

4. Find the area of a sector of a circle in inches whose arc length is 14 inches and whose radius is 2/3 feet.

ANSWERS:

1. 3,000 radians per minute

2. 1,500 feet

3. 24π square inches

4. 56 square inches

MILS

The *mil* is a unit of small angular measurement, which is not widely used but has some military applications in ranging and sighting. The *mil* is defined in two ways:

1. *As 1/6,400 of the circumference of a circle.*

2. *As the angle subtended by an object 1 unit long, perpendicular to the line of sight, at a distance of 1,000 units.*

From definition 1 we can see that since

$$360^\circ = 6{,}400 \text{ mils}$$

then

$$1^\circ = \frac{6{,}400}{360} \text{ mils}$$

$$= \frac{160}{9} \text{ mils}$$

$$= 17.78 \text{ mils (rounded)}$$

Also, since

$$6{,}400 \text{ mils} = 360^\circ$$

then

$$1 \text{ mil} = \frac{360^\circ}{6{,}400}$$

$$= \frac{9^\circ}{160}$$

$$= 0.05625^\circ$$

EXAMPLE: Convert 240 mils to degrees.

SOLUTION:

$$1 \text{ mil} = \frac{9^\circ}{160}$$

$$240 \text{ mils} = 240 \times 1 \text{ mil}$$

$$= 240 \times \frac{9^\circ}{160}$$

$$= \frac{27^\circ}{2}$$

$$= 13.5^\circ$$

EXAMPLE: Convert 27° to mils.

SOLUTION:

$$1^\circ = \frac{160}{9} \text{ mils}$$

$$27^\circ = 27 \times 1^\circ$$

$$= 27 \times \frac{160}{9} \text{ mils}$$

$$= 480 \text{ mils}$$

Since

$$1 \text{ mil} = \frac{9^\circ}{160}$$

and

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

then

$$1 \text{ mil} = \frac{9}{160} \times 1^\circ$$

$$= \frac{9}{160} \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{\pi}{3{,}200} \text{ radians}$$

$$= 0.00098 \text{ radians (rounded)}$$

We see that *1 mil is approximately 0.001* or *1/1,000 radians*. We also see that *1 radian ≈ 1,000 mils.*

EXAMPLE: Convert 25 mils to an approximate radian measure.

SOLUTION:

$$1 \text{ mil} \approx \frac{1}{1{,}000} \text{ radians}$$

$$25 \text{ mils} = 25 \times 1 \text{ mil}$$

$$\approx 25 \times \frac{1}{1{,}000} \text{ radians}$$

$$\approx \frac{25}{1{,}000} \text{ radians}$$

$$\approx 0.025 \text{ radians}$$

EXAMPLE: Convert 6.48 radians to an approximate measurement in mils.

SOLUTION:

$$1 \text{ radian} \approx 1{,}000 \text{ mils}$$

$$\begin{aligned} 6.48 \text{ radians} &= 6.48 \times 1 \text{ radian} \\ &\approx 6.48 \times 1{,}000 \text{ mils} \\ &\approx 6{,}480 \text{ mils} \end{aligned}$$

Referring to figure 3-8, when an angle, θ, subtended by an arc, s, is very small and the radius, r, is large, the chord, c, is almost equal to the arc, s.

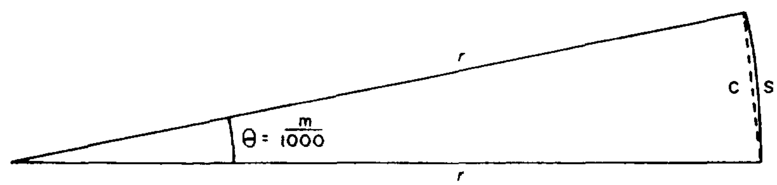

Figure 3-8.—Relationship of chord and arc.

The formula for the length of arc of a circle, as previously stated, is

$$s = r\theta$$

where θ is in radian measurement.

If the measurement of the arc is made in mils, we must divide the mil measure by 1,000 to obtain the radian measure. Since,

$$1 \text{ mil} = \frac{1}{1{,}000} \text{ radians (approximately)}$$

then

$$m \text{ mils} = \frac{m}{1{,}000} \text{ radians}$$

So,

$$\begin{aligned} s &= r\left(\frac{m}{1{,}000}\right) \\ &= \frac{rm}{1{,}000} \end{aligned}$$

Now, since the chord, c, in figure 3-8, is approximately equal to the arc, s, then

$$c = \frac{rm}{1{,}000}$$

Now consider definition 2. If

$$r = 1{,}000 \text{ yds}$$

and

$$m = 1 \text{ mil}$$

then

$$\begin{aligned} c &= \frac{rm}{1{,}000} \\ &= \frac{1{,}000 \times 1}{1{,}000} \\ &= 1 \text{ yard} \end{aligned}$$

We also know that the arc, s, is approximately equal to 1 yard since $s \approx c$.

The military uses the fact that a mil subtends a yard at a distance of 1,000 yards for quick computations in the field.

EXAMPLE: Find the length of a target if, at a right angle to the line of sight, it subtends an angle of 15 mils at a range of 4,000 yards.

SOLUTION:

$$\begin{aligned} c &= \frac{rm}{1{,}000} \\ &= \frac{4{,}000 \times 15}{1{,}000} \\ &= 60 \text{ yards} \end{aligned}$$

EXAMPLE: A building known to be 80 feet long and perpendicular to the line of sight subtends an angle of 100 mils. What is the approximate range to the building?

SOLUTION:

Since

$$c = \frac{rm}{1{,}000}$$

then

$$r = \frac{1{,}000c}{m}$$

$$= \frac{1{,}000 \times 80}{100}$$

$$= 800 \text{ feet}$$

PRACTICE PROBLEMS:

1. Convert 3,456 mils to degrees.
2. Convert 12 degrees to mils.
3. Convert 27,183 mils to an approximate radian measure.
4. Convert 431 radians to an approximate measurement in mils.
5. A tower 500 feet away subtends a vertical angle of 250 mils. What is the height of the tower?
6. If points X and Y are 48 yards apart and are 4,000 yards from an observer, how many mils do they subtend?

ANSWERS:

1. 194.4°
2. 213.3 mils
3. 27.183 radians

4. 431,000 mils

5. 125 feet

6. 12 mils

PROPERTIES OF RIGHT TRIANGLES

Mathematics, Volume 1, contains information on the trigonometric ratios and other properties of triangles. This section restates some of the properties of right triangles for review and reference.

PYTHAGOREAN THEOREM

The *Pythagorean theorem* states that in a right triangle, the square of the length of the hypotenuse (longest side) is equal to the sum of the squares of the lengths of the other two sides. In the right triangle shown in figure 3-9, this relationship is expressed as

$$r^2 = x^2 + y^2$$

where r is the length of the hypotenuse and x and y are the lengths of the other two sides.

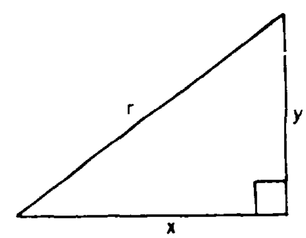

Figure 3-9.—Pythagorean relationship.

This relationship is useful in solving many problems and in developing trigonometric concepts.

EXAMPLE: In figure 3-10, what is the length of the hypotenuse of the right triangle if the lengths of the other two sides are 3 and 4?

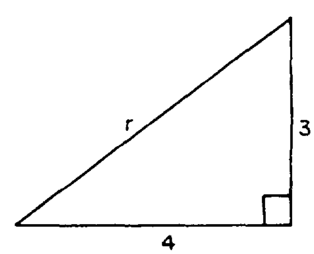

Figure 3-10.—Right triangle with hypotenuse unknown.

SOLUTION:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 4^2 + 3^2 \\ &= 16 + 9 \\ &= 25 \end{aligned}$$

So,

$$\begin{aligned} r &= \sqrt{25} \\ &= 5 \end{aligned}$$

NOTE: We will use the positive value of the square root since we are dealing with lengths.

EXAMPLE: Figure 3-11 shows a right triangle with a hypotenuse equal to 40 and one of the other sides equal to 10. What is the length of the remaining side?

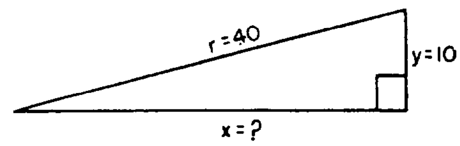

Figure 3-11.—Right triangle with one side unknown.

SOLUTION:

$$r^2 = x^2 + y^2$$

or

$$\begin{aligned} x^2 &= r^2 - y^2 \\ &= 40^2 - 10^2 \\ &= 1{,}600 - 100 \\ &= 1{,}500 \end{aligned}$$

So,

$$\begin{aligned} x &= \sqrt{1{,}500} \\ &= 38.7 \text{ (rounded)} \end{aligned}$$

SIMILAR RIGHT TRIANGLES

Another relationship of right triangles that is useful in trigonometry concerns *similar triangles*. Whenever the angles of one triangle are equal to the corresponding angles in another triangle, the two triangles are said to be *similar*.

For example, right triangle *A* in figure 3-12 is similar to right triangle *B*. Since the two triangles are similar by definition, the following proportions involving the lengths of the corresponding sides are true:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

This relationship can be used to find the lengths of unknown sides in similar triangles.

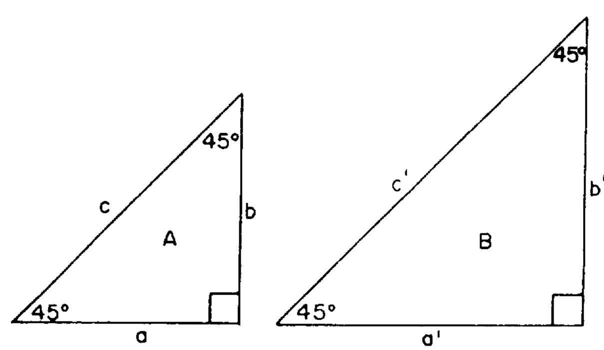

Figure 3-12.—Similar triangles.

EXAMPLE: Assume right triangles A and B in figure 3-13 are similar with lengths as shown. Find the lengths of sides b' and c'.

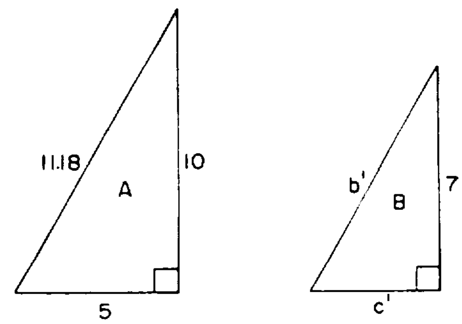

Figure 3-13.—Similar triangles, solution example.

SOLUTION:

Since

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

then

$$\frac{10}{7} = \frac{11.18}{b'} = \frac{5}{c'}$$

Side b' can be solved for using the first two ratios:

$$\frac{10}{7} = \frac{11.18}{b'}$$

So,

$$b' = \frac{11.18 \times 7}{10}$$

$$= \frac{78.26}{10}$$

$$= 7.826$$

Side c' can be solved for using the first and third ratios:

$$\frac{10}{7} = \frac{5}{c'}$$

So,

$$c' = \frac{5 \times 7}{10}$$

$$= 3.5$$

Recall from plane geometry that *the sum of the interior angles of any triangle is equal to 180°*. Using this fact, we can assume that two triangles are similar if two angles of one are equal to two angles of the other. The remaining angle in any triangle must be equal to 180° minus the sum of the other two angles.

If an acute angle of one right triangle is equal to an acute angle of another right triangle, the triangles are similar because the right angles in the two triangles are also equal to each other.

Hence, if θ is one of the acute angles in a right triangle, then $(90° - \theta)$ is the other acute angle, such that

$$90° + \theta + (90° - \theta) = 180°$$

Therefore, *two right triangles are similar if an acute angle of one triangle is equal to an acute angle of the other triangle.*

Many practical uses of trigonometry are based on the fact that two right triangles are similar if an acute angle of one triangle is equal to an acute angle of the other triangle.

In figure 3-14 triangle A is similar to triangle B since an acute angle in triangle A is equal to an acute angle in triangle B. Since triangle A is similar to triangle B, then

$$\frac{x}{x'} = \frac{y}{y'} = \frac{r}{r'}$$

Interchanging terms in the proportions gives

$$\frac{x}{y} = \frac{x'}{y'}$$

$$\frac{y}{r} = \frac{y'}{r'}$$

and

$$\frac{x}{r} = \frac{x'}{r'}$$

which are considered among the main principles of numerical trigonometry.

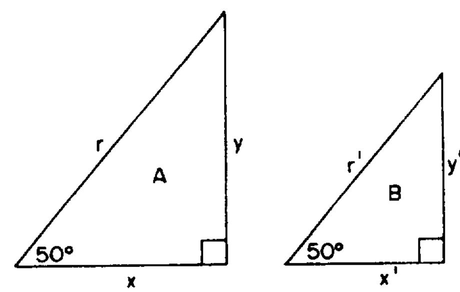

Figure 3-14.—Similar right triangles.

PRACTICE PROBLEMS:

Refer to figure 3-15 in solving the following problems:

1. Use the Pythagorean theorem to calculate the unknown length in triangle A.

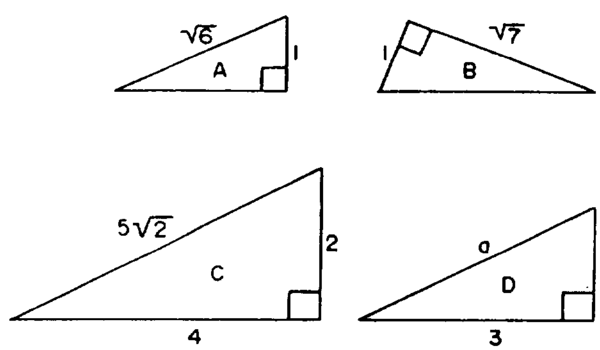

Figure 3-15.—Triangles for practice problems.

2. Use the Pythagorean theorem to calculate the unknown length in triangle *B*.

3. Triangles *C* and *D* are similar triangles. Find the length of sides *a* and *b* in triangle *D*.

ANSWERS:

1. $\sqrt{5}$

2. $2\sqrt{2}$

3. $a = 15\sqrt{2}/4$, $b = 3/2$

TRIGONOMETRIC RATIOS, FUNCTIONS, AND TABLES

The properties of triangles given in the previous section provide a means for solving many practical problems. Certain practical problems, however, require knowledge of right triangle relationships other than the Pythagorean theorem or the relationships of similar triangles before solutions can be found.

For example, the following two problems require additional knowledge:

1. Find the values of the unknown sides and angles in a right triangle when the values of one side and one acute angle are given.

2. Find the value of the unknown side and the values of the angles in a right triangle when two sides are known.

The additional relationships between the sides and angles of a right triangle are called *trigonometric ratios*. These ratios were introduced in *Mathematics*, Volume 1, and are reviewed in the following paragraphs. The basic foundations of trigonometry rest upon these ratios and their associated trigonometric functions.

TRIGONOMETRIC RATIOS AND FUNCTIONS

The sides of a right triangle form six ratios. In figure 3-16 we will use the acute angle θ and the three sides x, y, and r two at a time to define the trigonometric ratios. These ratios and the trigonometric functions associated with each ratio are listed as follows:

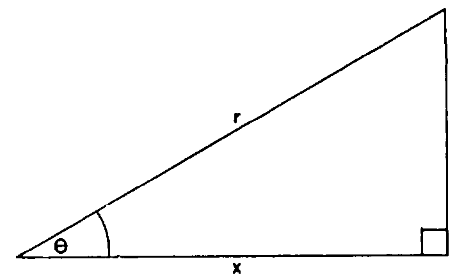

Figure 3-16.—Right triangle for determining ratios.

$$\text{the sine of } \theta = \frac{y}{r}, \text{ written } \sin \theta$$

$$\text{the cosine of } \theta = \frac{x}{r}, \text{ written } \cos \theta$$

$$\text{the tangent of } \theta = \frac{y}{x}, \text{ written } \tan \theta$$

$$\text{the cotangent of } \theta = \frac{x}{y}, \text{ written } \cot \theta$$

$$\text{the secant of } \theta = \frac{r}{x}, \text{ written } \sec \theta$$

$$\text{the cosecant of } \theta = \frac{r}{y}, \text{ written } \csc \theta$$

The trigonometric functions of a right triangle are remembered easier by the convention of naming the sides. Refer to figure 3-17. The side of length y is called the side *opposite* angle θ, the side of length x is called the side *adjacent* to angle θ, and the side of length r is called the *hypotenuse*. Using this terminology causes the six trigonometric functions to be defined as:

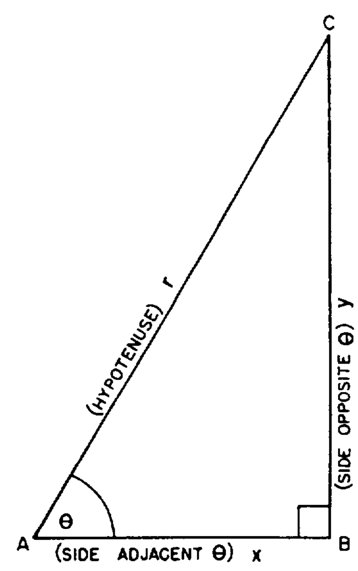

Figure 3-17.—Names of sides of a right triangle.

$$\sin \theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \theta = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot \theta = \frac{x}{y} = \frac{\text{adjacent}}{\text{opposite}}$$

$$\sec \theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\csc \theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite}}$$

Remember that the six trigonometric ratios apply only to the acute angles of a right triangle.

EXAMPLE: Give the values of the trigonometric functions of the angle in the right triangle for figure 3-18, view A.

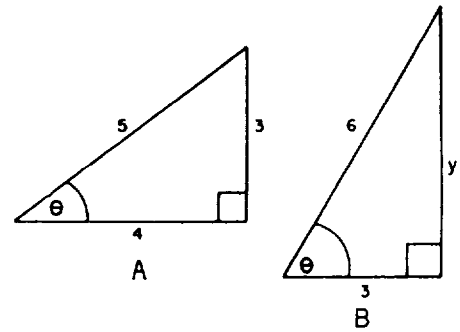

Figure 3-18.—Practice triangles.

SOLUTION:

$$\sin \theta = \frac{y}{r} = \frac{3}{5} = 0.6$$

$$\cos \theta = \frac{x}{r} = \frac{4}{5} = 0.8$$

$$\tan \theta = \frac{y}{x} = \frac{3}{4} = 0.75$$

$$\cot \theta = \frac{x}{y} = \frac{4}{3} = 1.33333 \text{ (rounded)}$$

$$\sec \theta = \frac{r}{x} = \frac{5}{4} = 1.25$$

$$\csc \theta = \frac{r}{y} = \frac{5}{3} = 1.66667 \text{ (rounded)}$$

EXAMPLE: Give the values of the trigonometric functions of the angle in the right triangle for figure 3-18, view B.

SOLUTION: Only two sides are given. To find the third side of the right triangle, use the Pythagorean theorem:

$$r^2 = x^2 + y^2$$

and

$$\begin{aligned} y^2 &= r^2 - x^2 \\ &= 6^2 - 3^2 \\ &= 36 - 9 \\ &= 27 \\ y &= \sqrt{27} \\ &= \sqrt{9 \cdot 3} \\ &= 3\sqrt{3} \end{aligned}$$

Now, using the values of x, y, and r, we find the values of the six trigonometric functions are as follows:

$$\sin\theta = \frac{y}{r} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} = 0.86603 \text{ (rounded)}$$

$$\cos\theta = \frac{x}{r} = \frac{3}{6} = \frac{1}{2} = 0.5$$

$$\tan\theta = \frac{y}{x} = \frac{3\sqrt{3}}{3} = \sqrt{3} = 1.73205 \text{ (rounded)}$$

$$\cot\theta = \frac{x}{y} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.57735 \text{ (rounded)}$$

$$\sec\theta = \frac{r}{x} = \frac{6}{3} = 2$$

$$\csc\theta = \frac{r}{y} = \frac{6}{3\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.15470 \text{ (rounded)}$$

TABLES OF TRIGONOMETRIC FUNCTIONS

Tables of trigonometric functions give the numerical values of the ratios of the sides of a right triangle that correspond to the trigonometric functions. Appendixes II and III are tables of trigonometric functions. These tables give values rounded to five decimal places of trigonometric functions for each minute from 0° to 90°. *Appendix II consists of tables of natural sines and cosines. Appendix III consists of tables of natural tangents and cotangents.*

For example, if we wanted to find sin 3° 25′, we would use appendix II, Natural Sines and Cosines, to first locate 3° on the first row of the table. Next, we would locate sin under 3° on the second row. Then, we would locate 25 along the first column of the table. Now, reading left to right across from 25 and from top to bottom under sin 3°, we find sin 3° 25′ = 0.05960. If we wanted to find cos 86° 35′, we would first locate 86° on the last row of the table. (The degrees on the top row range from 0° to 44°, and the degrees on the last row range from 45° to 90°.) Next, we would locate cos above 86° on the next to the last row. Then, we would locate 35 along the last column of the table. Now, reading right to left across from 35 and from bottom to top above cos 86°, we find cos 86° 35′ = 0.05960. Note that sin 3° 25′ = 0.05960 = cos 86° 35′. The reason for this will be discussed in chapter 4.

The tables in appendix III, Natural Tangents and Cotangents, are arranged in the same format as the tables in appendix II and are used in the same way. NOTE: Scientific calculators will give you the same values rounded to five decimal places as supplied in the tables in appendixes II and III.

Most tables list the sine, cosine, tangent, and cotangent of angles from 0° to 90°. Very few give the secant and cosecant since these functions of an angle are seldom used. When needed, they may be found from the values of the sine and cosine as follows:

$$\sec\theta = \frac{r}{x} = \frac{1}{\frac{x}{r}} = \frac{1}{\cos\theta}$$

and

$$\csc\theta = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\sin\theta}$$

Hence, the reciprocal of the secant function is the cosine function, and the reciprocal of the cosecant function is the sine function.

The tangent and cotangent functions may also be expressed in terms of the sine and cosine functions as follows:

$$\tan\theta = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin\theta}{\cos\theta}$$

and

$$\cot\theta = \frac{x}{y} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{\cos\theta}{\sin\theta}$$

In addition, the cotangent function may be determined as the reciprocal of the tangent function as follows:

$$\cot\theta = \frac{x}{y} = \frac{1}{\frac{y}{x}} = \frac{1}{\tan\theta}$$

NOTE: These relationships are the fundamental trigonometric identities that will be used extensively in solving more complex identities in chapter 6.

USE OF TRIGONOMETRIC RATIOS AND FUNCTIONS

The trigonometric ratios and trigonometric functions furnish powerful tools for use in problem solving of right triangles. Finding the remaining parts of a right triangle is possible if, in addition to the right angle, the length of one side and the length of any other side or the value of one of the acute angles is known.

EXAMPLE: Find the length of side y in figure 3-19, view A.

SOLUTION: We can use

$$\tan \theta = \frac{y}{x}$$

since we know one side and one angle. Thus,

$$\tan 35° = \frac{y}{20}$$

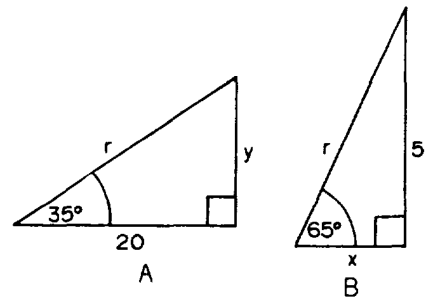

Figure 3-19.—Practical use of ratios.

From appendix III (or calculator), we find that

$$\tan 35° = 0.70021$$

So,

$$0.70021 = \frac{y}{20}$$

$$y = (0.70021)(20)$$

$$= 14.0042$$

We could have also used cos θ, cot θ, or sec θ to find side y.

EXAMPLE: Find the value of r in figure 3-19, view B.

SOLUTION:

$$\sin \theta = \frac{y}{r}$$

$$\sin 65° = \frac{5}{r}$$

$$r = \frac{5}{\sin 65°}$$

$$r = \frac{5}{0.90631}$$

$$= 5.51688$$

We could have also used csc θ to find side y.

PRACTICE PROBLEMS:

Refer to figure 3-20 in working problems 1 through 4.

1. Find the values of the trigonometric functions of angle θ for the right triangle in view A.

2. Find the value of side y in view B using the sine function.

3. Find the value of side x in view C using the cosine function.

4. Find the value of side y in view D using the tangent function.

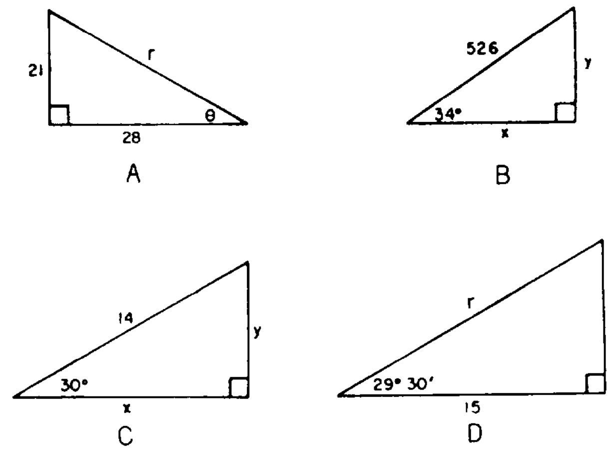

Figure 3-20.—Triangles for practice problems.

ANSWERS:

1. $\sin \theta = 21/35 = 3/5 = 0.6$

   $\cos \theta = 28/35 = 4/5 = 0.8$

   $\tan \theta = 21/28 = 3/4 = 0.75$

   $\cot \theta = 28/21 = 4/3 = 1.33333$

   $\sec \theta = 35/28 = 5/4 = 1.25$

   $\csc \theta = 35/21 = 5/3 = 1.66667$

2. 294.13394

3. 12.12442

4. 8.48655

SUMMARY

The following are the major topics covered in this chapter:

1. **Terminology:**

   Radius vector—The line that is rotated to generate an angle.

   Initial position—The original position of the radius vector.

   Terminal position—The final position of the radius vector.

   Positive angle—The angle generated by rotating the radius vector counterclockwise from the initial position.

   Negative angle—The angle generated by rotating the radius vector clockwise from the initial position.

2. **Degrees:** The degree system is the most common system of angular measurement. In this system a complete revolution is divided into 360 equal parts called *degrees.*

   $$1 \text{ revolution} = 360^\circ$$

   $$1^\circ = 60'$$

   $$1' = 60''$$

   For convenience, the 360° are divided into four equal parts of 90° each called *quadrants.*

   If $0^\circ < \theta < 90^\circ$, then θ is in quadrant I.

   If $90^\circ < \theta < 180^\circ$, then θ is in quadrant II.

   If $180^\circ < \theta < 270^\circ$, then θ is in quadrant III.

   If $270^\circ < \theta < 360^\circ$, then θ is in quadrant IV.

   If $\theta > 360^\circ$, then θ lies in the same quadrant as $\theta - n(360^\circ)$, where $n = 1, 2, 3, \ldots$ and $n(360^\circ) < \theta$.

3. **Radians:** An even more fundamental method of angular measurement involves the *radian.* A *radian* is defined as an angle that, if its vertex is placed at the center of a circle, intercepts an arc equal in length to the radius of the circle.

   $$2\pi \text{ radians} = 360^\circ$$

   $$\pi \text{ radians} = 180^\circ$$

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$1^\circ = \frac{\pi}{180^\circ} \text{ radians}$$

The radian measure of an angle, θ, is the ratio of the length of the arc, s, it subtends to the length of the radius vector, r, of the circle in which it is the central angle or

$$\theta = \frac{s}{r}$$

4. **Other frequently used relationships between radians and degrees:**

Radians	Degrees
$\pi/6$	30
$\pi/4$	45
$\pi/3$	60
$\pi/2$	90
π	180
$3\pi/2$	270
2π	360

5. **Length of arc:**

$$s = \theta r$$

where θ represents the number of radians in a central angle, r the length of the radius of the circle, and s the length of the intercepted arc.

6. **Angular velocity:**

$$\omega = \frac{\theta}{t}$$

where θ is measured in radians and t is the unit time.

7. **Linear velocity:**

$$v = \frac{d}{t}$$

where d is the distance and t is the unit time.

$$v = r\omega$$

where r is the radius and ω is the angular velocity.

8. **Area of a sector of a circle:**

$$A = \frac{1}{2}r^2\theta$$

where θ is expressed in radians.

$$A = \frac{1}{2}rs$$

where r is the radius and s is the arc length.

9. **Mils:** The *mil* is a unit of small angular measurement that has military applications. The *mil* is defined as follows:

   1. 1/6,400 of the circumference of a circle.

   $$360° = 6{,}400 \text{ mils}$$

   $$1° = \frac{160}{9} \text{ mils}$$

   $$1 \text{ mil} = \frac{9°}{160}$$

   2. The angle subtended by an object 1 unit long, perpendicular to the line of sight, at a distance of 1,000 units.

   $$1 \text{ mil} \approx \frac{1}{1{,}000} \text{ radians}$$

   $$1 \text{ radian} \approx 1{,}000 \text{ mils}$$

10. **Pythagorean theorem:** The *Pythagorean theorem* states that in a right triangle, the square of the length of the hypotenuse, r, is equal to the sum of the squares of the lengths of the other two sides, x and y, or

$$r^2 = x^2 + y^2$$

11. **Similar triangles:** Whenever the angles of one triangle are equal to the corresponding angles in another triangle, the two triangles are said to be *similar* and the following proportions involving the lengths of their corresponding sides are true:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

12. **Similar right triangles:** Two right triangles are similar if an acute angle of one triangle is equal to an acute angle of the other triangle. The following proportions involving the lengths of their corresponding sides are true:

$$\frac{x}{x'} = \frac{y}{y'} = \frac{r}{r'}$$

13. **Trigonometric ratios and functions:**

$$\sin\theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot\theta = \frac{x}{y} = \frac{\text{adjacent}}{\text{opposite}}$$

$$\sec\theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\csc\theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite}}$$

14. **Tables of trigonometric functions:** Tables of trigonometric functions give the numerical values of the ratios of the sides of a right triangle that correspond to the trigonometric functions. Appendix II consists of tables of natural sines and and cosines. Appendix III consists of tables of natural tangents and cotangents.

ADDITIONAL PRACTICE PROBLEMS

1. In which quadrant is the angle $5{,}370°$?

2. Find the radian measure of the central angle in a circle with radius π inches if the angle subtends an arc of $3\pi/5$ inches.

3. Express $4{,}320°$ in radians, using π in the answer.

4. Express $11\pi/12$ in degrees.

5. If the length of the radius of a circle is 5 meters, find the length of arc subtended by a central angle with measure π radians.

6. Kim and Tom are riding on a Ferris wheel. Kim observes that it takes 30 seconds to make a complete revolution. Their seat is 35 feet from the axle of the wheel.

   a. What is their angular velocity in radians per second?

   b. What is their linear velocity in feet per minute?

7. Find the area of a sector of a circle if its central angle is $45°$ and the diameter of the circle is 28 centimeters.

8. Convert 17 7/9 mils to degrees.

9. Convert 3.6 degrees to mils.

10. Convert 9/5 mils to an approximate radian measure.

11. Convert 0.00145 radians to an approximate measurement in mils.

12. An airplane with a wing span of 84 feet is flying toward an observer. What is the distance of the plane from the observer when the plane subtends 7 mils?

13. The length of the hypotenuse of a right triangle is 17, and the length of one of the other sides is 8. What is the length of the remaining side?

14. Assume similar right triangles A and B have sides $x, y, r,$ and x', y', r', respectively. If $x = 6$, $y = 8$, $r = 10$, and $y' = 1/2$, what are the values of x' and r'?

15. Find the values of the trigonometric functions θ of in a right triangle if the hypotenuse is 25 and the side adjacent to θ is 24.

16. If in a right triangle one of the acute angles is $56° \; 17'$ and the hypotenuse is 10, what are the lengths of the other two sides?

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 4th

2. 3/5

3. 24π

4. $165°$

5. 5π meters

6. a. $\pi/15$ radians per second

   b. 140π feet per minute

7. $49\pi/2$ square centimeters

8. $1°$

9. 64 mils

10. 0.0018 radians

11. 1.45 mils

12. 12,000 feet

13. 15

14. $x' = 3/8$

    $z' = 5/8$

15. $\sin\theta = 7/25 = 0.28$

    $\cos\theta = 24/25 = 0.96$

    $\tan\theta = 7/24 = 0.29167$ (rounded)

    $\cot\theta = 24/7 = 3.42857$ (rounded)

    $\sec\theta = 25/24 = 1.04167$ (rounded)

    $\csc\theta = 25/7 = 3.57143$ (rounded)

16. 5.5509 and 8.3179